# ON THE STABILITY OF ROTATION OF A SOLID WITH AN ELLIPSOIDAL CAVITY FILLED WITH LIQUID

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In the works of Joukowski [1], Sludsky [2], Hough [3] and Poincaré [4] it was shown that the motion of a solid with an ellipsoidal cavity completely filled with ideal liquid can be described by ordinary differential equations. The same fact of reducibility of the equations of motion to ordinary differential equations (for an ellipsoidal cavity) has been proved in [5].

In the article by Rumiantsev [6] the sufficient conditions of stability have been obtained.

In the present article the reducibility of the equations of motion is used for the investigation of the stability of motion of a solid having a cavity in the form of a triaxial ellipsoid, completely filled with ideal liquid, which is in the state of uniform vortex motion. Let us note that Chetaev [7] has given the solution of a similar problem in the case of the irrotational motion of the liquid.

Let $Ox_1y_1z_1$ be a fixed coordinate system with the origin at a fixed point $O$ in the solid, $z_1$ being directed vertically upwards, and let $Oxyz$ be a moving coordinate system whose axes coincide with the principal axes of inertia of the solid at the point $O$; the equation of the surface of the cavity filled with liquid in the $xyz$ system is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1 \qquad (1)$$

The motion of the liquid which occupies the cavity (I), can be described [2,3] by the formulas

$$v_x = \frac{\partial\varphi}{\partial x} + \omega_2 z - \omega_3 y, \qquad v_y = \frac{\partial\varphi}{\partial y} + \omega_3 x - \omega_1 z, \qquad v_z = \frac{\partial\varphi}{\partial z} + \omega_1 y - \omega_2 x \tag{2}$$

where $v_x$, $v_y$, $v_z$ designate, respectively, the projections of the absolute velocity vector $\mathbf{v}$ of the liquid on the moving coordinate axes; $\varphi(x, y, z, t)$ is a harmonic function of the coordinates; $\omega_1$, $\omega_2$, $\omega_3$ are functions of time $t$ only. The function $\varphi$ can be written in the explicit form [6]. Let us designate the projections of the vector $\omega$ of the instantaneous angular velocity of the solid on the $x$, $y$, $z$ axes by $p$, $q$, $r$, and the direction cosines of the fixed axis $z_1$ with respect to the moving axes, by $\gamma_1$, $\gamma_2$, $\gamma_3$.

The motion of the system shell-liquid can be described [6] by the following three groups of equations, in which the first group corresponds to the theorem of the moment of momentum, the second to the Helmholtz equation of vortex motion and the third to Poisson's equation for the direction cosines

$$A\frac{dp}{dt} + A_2\frac{d\omega_1}{dt} + (C-B)\,qr + C_2 q\omega_3 - B_2 r\omega_2 = R\gamma_2$$

$$B\frac{dq}{dt} + B_2\frac{d\omega_2}{dt} + (A-C)\,rp + A_2 r\omega_1 - C_2 p\omega_3 = -R\gamma_1 \qquad (R = Mgz^\circ)$$

$$C\frac{dr}{dt} + C_2\frac{d\omega_3}{dt} + (B-A)\,pq + B_2 p\omega_2 - A_2 q\omega_1 = 0 \tag{3}$$

$$\frac{d\omega_1}{dt} = 2a^2\left(\frac{r\omega_2}{a^2+b^2} - \frac{q\omega_3}{c^2+a^2}\right) - 2\,\omega_2\omega_3\frac{a^2(c^2-b^2)}{(a^2+b^2)(a^2+c^2)}, \qquad \frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3$$

$$\frac{d\omega_2}{dt} = 2b^2\left(\frac{p\omega_3}{b^2+c^2} - \frac{r\omega_1}{a^2+b^2}\right) - 2\,\omega_3\omega_1\frac{b^2(a^2-c^2)}{(b^2+c^2)(b^2+a^2)}, \qquad \frac{d\gamma_2}{dt} = p\gamma_3 - r\gamma_2$$

$$\frac{d\omega_3}{dt} = 2c^2\left(\frac{q\omega_1}{c^2+a^2} - \frac{p\omega_2}{b^2+c^2}\right) - 2\,\omega_1\omega_2\frac{c^2(b^2-a^2)}{(c^2+a^2)(c^2+b^2)}, \qquad \frac{d\gamma_3}{dt} = q\gamma_1 - p\gamma_2$$

Here, $M$ is the total mass of the solid and liquid, $z^\circ$ is the coordinate of the centroid of the system. The sums of the moments of inertia of the solid $A_1$, $B_1$, $C_1$ and of the equivalent solid in Joukowski's sense [1] $A_1'$, $B_1'$, $C_1'$, with respect to the moving axes, are designated by $A$, $B$, $C$

$$A = A_1 + A_1', \qquad B = B_1 + B_1', \qquad C = C_1 + C_1'$$

$$A_1' = \frac{m_2}{2}\frac{(b^2-c^2)^2}{b^2+c^2} + m_2 z_0^2, \quad B_1' = \frac{m_2}{5}\frac{(c^2-a^2)^2}{c^2+a^2} + m_2 z_0^2, \quad C_1' = \frac{m_2}{5}\frac{(a^2-b^2)^2}{a^2+b^2} \tag{4}$$

$$m_2 = 4/3\,\pi\rho abc$$

The differences between the corresponding moments of inertia of the liquid and of the equivalent solid are designated by $A_2$, $B_2$, $C_2$

$$A_2 = \frac{4m_2}{5}\frac{b^2c^2}{b^2+c^2}, \qquad B_2 = \frac{4m_2}{5}\frac{a^2c^2}{a^2+c^2}, \qquad C_2 = \frac{4m_2}{5}\frac{a^2b^2}{a^2+b^2} \tag{5}$$

The complete system of equations (3) allows a particular solution

$$p=0,\quad q=0,\quad r=\omega;\quad \omega_1=0,\quad \omega_2=0,\quad \omega_3=\omega;\quad \gamma_1=0,\quad \gamma_2=0,\quad \gamma_3=1$$

Let us investigate the stability of motion with respect to the variables $p$, $q$, $r$, $\omega_1$, $\omega_2$, $\omega_3$, $\gamma_1$, $\gamma_2$, $\gamma_3$. We obtain the equations of the perturbed motion by setting

$$r=\omega+\xi,\qquad \omega_3=\omega+\eta,\qquad \gamma_3=1+\zeta$$

in the perturbed motion and retaining previous designations for the other variables. We will not write down the variational equations, since obtaining them is elementary. The last of the variational equations in each group, obtained respectively from the systems (3), have the obvious solutions

$$\xi=\text{const},\qquad \eta=\text{const},\qquad \zeta=\text{const}$$

We will look for the solution of the remaining six variational equations in the form $e^{i\lambda t}$. The characteristic equation will be cubic with respect to $\lambda_1 = \lambda^2/\omega^2$

$$\begin{aligned}
&\lambda^6 AB-\lambda^4\omega^2\Big[AB(1+\mu\nu)+(C+C_2-B-\nu B_2)(C+C_2-A-\mu A_2)+\\
&+B_2B(\nu-1)\nu+A_2A(\mu-1)\mu-\frac{R}{\omega^2}(A+B)\Big]-\lambda^2\omega^4\Big[\frac{R}{\omega^2}\mu\nu(A+B)+\\
&+\frac{R}{\omega^2}\mu(\mu-1)A_2+\frac{R}{\omega^2}\nu(\nu-1)B_2+A_2A\mu(1-\mu)+B_2B\nu(1-\nu)-AB\mu\nu-\\
&-\left(\frac{R}{\omega^2}+A+\mu A_2-C-C_2\right)\left(\frac{R}{\omega^2}+B+\nu B_2-C-C_2\right)-\\
&-\mu\nu(A+A_2-C-C_2)(B+B_2-C-C_2)\Big]-\\
&-\mu\nu\omega^6\left(\frac{R}{\omega^2}+A+A_2-C-C_2\right)\left(\frac{R}{\omega^2}+B+B_2-C-C_2\right)=0
\end{aligned}\tag{6}$$

$$\left(\mu=\frac{2a^2}{a^2+c^2},\quad \nu=\frac{2b^2}{b^2+c^2}\right)$$

For stability it is necessary that all $\lambda$ be real, i.e. that all $\lambda_1$ be non-negative. The conditions of $\lambda_1$ being non-negative provide the necessary conditions of stability. The latter become rather cumbersome, so without writing them down, let us point out a condition of instability which follows directly from the form of the constant term. If

$$\frac{R}{\omega^2}+A+A_2-C-C_2,\qquad \frac{R}{\omega^2}+B+B_2-C-C_2$$

have different signs, then there must exist at least one root $\lambda_1 < 0$ and consequently the motion is unstable. It has been proved in the article [6] that the conditions

$$\frac{R}{\omega^2}+A+A_2-C-C_2<0,\qquad \frac{R}{\omega^2}+B+B_2-C-C_2<0$$

are the sufficient conditions of stability. In the case

$$\frac{R}{\omega^2} + A + A_2 - C - C_2 > 0, \qquad \frac{R}{\omega^2} + B + B_2 - C - C_2 > 0 \tag{7}$$

both stable and unstable motions are possible, depending on the actual magnitudes of these expressions. The investigation of the roots under the conditions (7) is difficult in a general case, but in each specific instance, when all the quantities except $\omega$ are given, one can look for the values of $\omega$ which will insure stability, turning directly to the characteristic equation (6).

In the case when $A_1 = B_1$, $a = b$, i.e. in the case of a symmetrical top with a cavity in the form of an ellipsoid of rotation, which has been considered in [5,8], the investigation can be somewhat simplified. If we introduce the variables

$$\mathrm{P} = p + iq, \qquad \Omega = \omega_1 + i\omega_2, \qquad \Gamma = \gamma_1 + i\gamma_2$$

the variational equations take the following form:

$$A\frac{d\mathrm{P}}{dt} = i\omega(C + C_2 - A - A_2\mu)\,\mathrm{P} - i\omega A_2(1-\mu)\,\Omega - iR\Gamma$$

$$\frac{d\Omega}{dt} = i\omega\mu(\mathrm{P} - \Omega), \qquad \frac{d\Gamma}{dt} = i\mathrm{P} - i\omega\Gamma$$

We look for the solution in the form $e^{i\lambda t}$. Thus we obtain the following characteristic equation:

$$A\lambda_1^3 + \lambda_1^2[A(2+\mu) + A_2\mu - C - C_2] + \lambda_1\left[\frac{R}{\omega^2} + A(1+2\mu) + 2A_2\mu - (C+C_2)(1+\mu)\right] + $$
$$+ \mu\left[\frac{R}{\omega^2} + A + A_2 - C - C_2\right] = 0 \qquad \left(\lambda_1 = \frac{\lambda}{\omega}\right)$$

This equation coincides with the one obtained in [5].

The stability condition is the condition of roots being real. It coincides with the condition obtained in [5,8].

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